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groups of eight lying just as they escaped from the oogonial sac and conspicuously oriented with respect to each other. The fact that groups of ten or of two are as regularly oriented, would refute the suggestion; but in order to prove that the phenomenon is the result of a stimulus acting after the eggs leave the oogonium, a group of them were transferred to a watch crystal and mixed with the point of a needle until their relative positions were entirely changed. But when they germinated the characteristic orientation with respect to each other was found to be without an exception.

The only apparent explanation of the group orientation is that of a diffusion gradient of some substance emanating from a growing spore, or of some substance being used up by it. A continuation of the investigation of this problem will be an attempt to discover a substance which can so affect the dividing nucleus of the egg cell that its unequal distribution on the sides of the cell will orient the axis of the spindle. The effect of bubbling carbon dioxide and oxygen through cultures should be tried as being the most probable factors involved.

The substance or condition originating in the activity of adjacent spores which has so powerful an effect in orienting the first cleavage plane and in determining which cell shall become the rhizoidal cell has no power to cause any chemotropism of the rhizoids after they are started. No rhizoid has been found to have its direction modified by the presence of other spores adjacent to it. In the absence of any light stimulus the rhizoids continue in the direction that they take originally from the spore.

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ON THE MOST GENERAL CLASS L OF FRÉCHET IN WHICH THE HEINE-BOREL-LEBESGUE THEOREM HOLDS TRUE

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Communicated by E. H. Moore, April 15, 1919

§1 A class L of Fréchet[1] is a set of elements such that (1) if P is an element of L and P_1, P_2, P_3 . . . is a countable[2] sequence of elements belonging to L then the statement that P is the limit of the sequence P_1, P_2, P_3, . . .

has a definite meaning and the question whether this statement is true or false has a determinate answer as soon as the element P and the sequence in question are themselves determined, (2) if the element P is the limit of the sequence $P_1, P_2, P_3, \ldots$ and $n_1, n_2, n_3, \ldots$ is an infinite sequence of positive integers such that $n_1<n_2<n_3 \ldots$, then P is also the limit of the sequence $P_{n_1}, P_{n_2}, P_{n_3}, \ldots$, (3) if P is an element of L, P is the limit of the sequence $P, P, P, \ldots$, whose elements all coincide with the element P. An element P is said to be a *limiting element* of a sub-class M of the class L if P is the limit of some infinite sequence of distinct elements belonging to M. The set M is said to be *compact* if every infinite set of distinct elements belonging to M has at least one limiting element. The totality of all the limiting elements of a given set M is called the derived set of M. A class S is a class L in which the derived set of every set is closed. An element P belonging to L is said to be *interior to* the sub-class M of the class L if M contains P and at least one element of every sequence of distinct elements that converges to P. A family G of sub-classes of a class L is said to *cover* a subclass M of the class L if every element of M is interior to some member of the family G. A set of elements M is said to possess the Heine-Borel property if for every *countably* infinite family G of sub-classes of L that covers M there is a finite sub-family of G that also covers M. A sub-set M of L is said to possess the Heine-Borel-Lebesgue property if for *every* family G of sub-classes of L that covers M there is a finite sub-family of G that covers M.

In a recent paper[3] Fréchet has shown that in order that, in a given class S, a point-set[4] M should have the Heine-Borel property it is necessary and sufficient that the set M should be closed and compact. He also shows that the same conditions are necessary and sufficient in order that a point set M in a class V[5] should have the Heine-Borel-Lebesgue property. The Heine-Borel Theorem or the Heine-Borel-Lebesgue Theorem is said to hold true in a given space L if in that particular space every closed and compact point-set has the Heine-Borel property or the Heine-Borel-Lebesgue property respectively. Fréchet points out that in order that the Heine-Borel-Lebesgue Theorem should hold true in a given class L it is necessary, but not sufficient, that the class L should be a class S; and sufficient, but not necessary, that it should be a class V. He raises the question as to what property it is *necessary and sufficient* that a class L should possess in order that the Heine-Borel-Lebesgue Theorem should hold true in that class. In the present paper I will exhibit one such property.

I will call a family G of point-sets a *monotonic family* if, for every two point-sets of the family G, one of them is a subset of the other one. A sub-class M of a class L will be said to have the property K in case it is true that for every monotonic family G of closed sub-classes of M there is at least one point which is common to all the members of the family G. A class S in which every compact subset has the property K will be called a class S^*.

Lemma 1. *If in a class L every set of points which contains the point P in its interior contains at least one point of the set M distinct from P, then the point P is a limiting element of the set M.*

Proof. If P were not a limiting element of M then it would be interior to $L - M + P$. But this set of elements contains no point of M distinct from P.

Theorem 1. *If, in a class S^*, M is a closed and compact set of points and β is a well-ordered sequence of point-sets such that M is covered by the family composed of all the members of β, then there exists a member g of the sequence β such that M is covered by the family composed of g together with all those members of β that precede g.*

Proof. Suppose there exists no such member g. Then for each member x of the sequence β let M_x denote the set of all points P belonging to M such that P is not in the interior of x or of any member of β that precedes x. For every x, M_x contains at least one point. For every x, M_x is closed. For suppose P is a limiting element of M_x. If P were not in M_x then it would necessarily be in the interior either of the point-set x or of some point-set of the sequence β that precedes x and therefore, by a lemma of Hedrick's,[6] this particular point-set would contain, in its interior, at least one point of M_x, which is contrary to the definition of M_x. It follows that the family composed of all M_x's for all members x of the sequence β is a monotonic family of closed point-sets. Hence there exists a point P_0 which is in every M_x. But P_0 is in the interior of some point-set x_0 of the sequence β. Hence P_0 is not in M_{x_0}. Thus the supposition that Theorem 1 is not true leads to a contradiction.

Theorem 2. *In order that the Heine-Borel-Lebesgue Theorem should hold true in a given class S it is necessary and sufficient that that class S should be also a class S^*.*

Proof. Suppose that in a given class S^* the closed and compact point-set M is covered by the infinite family G of point-sets. The members of the family G can[7] be arranged in a well-ordered sequence β. By Theorem 1, there exists at least one member g of β such that M is covered by the family composed of g and all those members of β that precede g, together with not more than a finite number of those elements of β that follow g. Let g_1 denote the first[8] such g and let $g_2, g_3, g_4, \ldots g_n$ denote a finite set of members following g_1 such that M is covered by $g_1, g_2, g_3, \ldots g_n$ together with all those members of β that precede g_1. There clearly cannot exist a member of β immediately preceding g_1. I will show that g_1 is the first member of β. Suppose it is not. Then if g_1 and all the succeeding members of the sequence β are removed, there remains a well-ordered sequence β_1 whose members are the remaining members of β arranged in exactly the same order as in the original sequence β. Let us now construct a third well-ordered sequence β_2 having for its first n members the point-sets $g_1, g_2, g_3, \ldots g_n$ arranged in the order indicated and having as its remaining members the members of β_1 arranged in the same order as in β_1. The point-set β_2 has no last member. The point-set M is covered by the family of point-sets composed of all the members of the sequence β_2 but there does not exist any member g of β_2 such that M is covered by the family composed of g together with all those members of β_2 that precede g. But this is contrary to Theorem 1. Thus the supposition that g_1 is not the first member of β has led to a contradiction. It follows that g_1 is the first member of β and that M is covered by the finite set of point-sets $g_1, g_2, g_3, \ldots g_n$. Thus the Heine-Borel-Lebesgue Theorem holds true in every class S^*.

Suppose now that the Heine-Borel-Lebesgue Theorem holds true in a given class S and that G is a monotonic family of closed, compact subsets of S. Let $\bar{g}$ denote any point-set of the family G. I will show that the members of G have at least one point in common. Suppose that this is not the case. Then, if P is a point of $\bar{g}$, there exists a closed point-set g_P of the family G that does not contain P. Hence, by Lemma 1, the point P is in the interior of some point-set R_P which contains no point of g_P. Let H denote the set of all R_P's for all points P of $\bar{g}$. By hypothesis, $\bar{g}$ is covered by a finite subfamily $R_{P_1}, R_{P_2}, R_{P_3}, \ldots R_{P_n}$ of the family H. But there exists an i $(1 \leqq i \leqq n)$ such that g_{P_i} is a subset of each of the point-sets $g_{P_1}, g_{P_2}, g_{P_3}, \ldots g_{P_n}$. It follows that no point of g_{P_i} is in any one of the sets $R_{P_1}, R_{P_2}, R_{P_3}, \ldots R_{P_n}$. Thus the supposition that there is no point common to all the members of G has led to a contradiction.

§2. I now raise the question whether it is not desirable to substitute, for Fréchet's definition of the word compact, a definition which is, for some spaces, (but not for spaces V) more restrictive than that of Fréchet. I will say that a monotonic family of point-sets is *proper* if there is no point that is common to all of its members. I will say that a set of points M is *compact in the new sense*[9] if for every proper monotonic family F of subsets of M there exists at least one point which is a limit point of every point-set of F.

Suppose that in a space L the infinite point-set N is compact in the new sense. The set N contains at least one countably infinite sequence of distinct points $P_1, P_2, P_3, \ldots$ For each n let t_n denote the point-set $P_n, P_{n+1}, P_{n+2}, \ldots$ The family of point-sets $t_1, t_2, t_3, \ldots$ is a proper monotonic family. Hence there exists a point P which is a limiting element of every one of these point-sets. It follows that if in a space L a point-set is compact in the new sense then it is also compact in the sense of Fréchet. That the converse is not true for every space S may be seen with the help of the following example.

Example. Let α be a well-ordered set of elements such that [10] (1) if K is any countable subset of the elements of α then there exists an element of α that follows all the elements of K, (2) if P is a given element of α the set of all those elements of α that precede P is countable. If a and b are two non-consecutive elements of α such that a precedes b then the set of all those elements of α which follow a and precede b will be called a segment. An element P of α will be said to be the *limit* of a countable sequence $P_1, P_2, P_3 \ldots$ of elements of α if, and only if, for every segment s that contans P there exists a positive integer δ_{PS} such that, for every n greater than δ_{PS}, P_n is in s. With respect to this conception of limit of a sequence, the elements of α evidently constitute a class S. The set M composed of all the elements of α is compact in the sense of Fréchet. It is not, however, compact in the new sense. For if for every element x of α, t_x denotes the set of all those elements of α which follow x then there exists no element which is a limiting element of every member of the proper monotonic family composed of all t_x's for all elements x of α. The set M, though closed and compact in the sense of Fréchet, does not possess the Heine-Borel-Lebesgue property.

It is easy to see that, in every class V of Fréchet, a point-set which is compact in the Fréchet sense is also compact in the new sense.[11]

By a proof in large part identical with the above proofs of Theorems 1 and 2, the truth of the following theorem may be established.

Theorem 3. *In a class S, in order that a point-set M should possess the Heine-Borel-Lebesgue property it is necessary and sufficient that M should be compact in the new sense and closed.*

[1] Fréchet, M., *Palermo, Rend. Circ. Mat.*, 22, 1906, (1–72).

[2] By a countable sequence is meant a sequence of the same order type as the sequence of positive integers arranged in the normal order.

[3] Fréchet, M., *Bul. sci. math., Paris*, **45**, 1917, (1–8). See also Chittenden, E. W., *Bull. Amer. Math. Soc., New York*, **25**, 1918, (60–65).

[4] In the present paper the elements of a class L will be called points.

[5] A Class V is a class L in which there exists a distance function. Cf. either of the above mentioned papers of Fréchet.

[6] Hedrick, E. R., *Trans. Amer. Math. Soc., New York*, **12**, 1911, (285–294). Cf. also Fréchet, loc. cit.

[7] For a proof that the elements of any set (whether countable or uncountable) can be arranged in a well-ordered sequence, see Zermelo, E., *Math. Ann., Leipzig*, **59**, 1904, (514) and **65**, 1908, (107–128). Zermelo assumes the truth of the well-known Zermelo Postulate. With regard to this postulate, see a recent paper by Ph. E. B. Jourdain, *Paris, C. R. Acad. Sci.*, **166**, 1918, (520 and 984).

[8] Every subset of a well-ordered sequence contains a first member, that is to say, a member that precedes every other member of that subset.

[9] It is clear that in every space S a point-set is compact in the new sense if and only if, it possesses the property K defined in §1 of the present paper.

[10] For a proof of the existence of a well-ordered sequence satisfying these two conditions cf. Hobson, E. W., *The Theory of Functions of a Real Variable*, Univ. Press, Cambridge, 1913, (177–181).

[11] Sometime after the manuscript of the present paper had left my hands I found that, in 1912, S. Janiszewski introduced an extended conception *of limit* and defined a point-set as "*parfaitement compact*, si de toute de ses éléments on peut extraire une suite du même type d'ordre et possédant un élémente limite." cf. J. éc. polytech., Paris, 16, 1912 (155). Compare also the example in §2 of the present paper with an example of Janiszewski's on page 167, loc. cit. It seems likely that in a class S a set is compact in my sense if and only if it is parfaitement compact in the sense of Janiszewski. I do not find however that Janiszewski has made any study of the Heine-Borel-Lebesgue Theorem in connection with his conception of compactness.